ON THE UNIQUENESS OF GIBBS STATES IN THE PIROGOV-SINAI THEORY*

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Abstract. We prove that, for low-temperature systems considered in the Pirogov-Sinai theory, uniqueness in the class of translation-periodic Gibbs states implies global uniqueness, i.e. the absence of any non-periodic Gibbs state. The approach to this infinite volume state is exponentially fast.

Key words: Gibbs state, Pirogov-Sinai theory, uniqueness, cluster (polymer) expansion.

Dedicated to the memory of Roland Dobrushin

1. Introduction

The problem of uniqueness of Gibbs states was one of R.L.Dobrushin's favorite subjects in which he obtained many classical results. In particular when two or more translation-periodic states coexist, it is natural to ask whether there might also exist other, non translation-periodic, Gibbs states, which approach asymptotically, in different spatial directions, the translation periodic ones. The affirmative answer to this question was given by R.L.Dobrushin with his famous construction of such states for the Ising model, using $\pm$ boundary conditions, in three and higher dimensions [D]. Here we consider the opposite situation: we will prove that in the regions of the low-temperature phase diagram where there is a unique translation-periodic Gibbs state one actually has *global uniqueness* of the limit Gibbs state. Moreover we show that, uniformly in boundary conditions, the finite volume probability of any local event tends to its infinite volume limit value exponentially fast in the diameter of the domain.

The first results concerning this problem in the framework of the Pirogov-Sinai theory [PS] were obtained by R.L.Dobrushin and E.A.Pecherski in [DP]. The Pirogov-Sinai theory describes the low-temperature phase diagram of a wide class of spin lattice models, i.e. it determines all their translation-periodic limit Gibbs states [PS], [Z]. The results of [DP], corrected and extended in [Sh], imply that, for any values of parameters at which the model has a unique ground state, the Gibbs state is unique for sufficiently small temperatures. But the closer these parameters are to the points with non unique ground state the smaller the temperature for which uniqueness of the Gibbs state is given by this method. Independently, an alternative method leading to similar results was developed in [M1,2].

The main difficulty in establishing the results of this type is due to the necessity of having sufficiently detailed knowledge of the partition function in a finite domain with an

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arbitrary boundary condition. This usually requires a detailed analysis of the geometry of the so called *boundary layer* produced by such a boundary condition (see [M1,2], [Sh]). Here we develop a new simplified approach to the problem. The simplification is achieved by transforming questions concerning the finite volume Gibbs measure with an arbitrary boundary conditions into questions concerning the distribution with a *stable* (in the sense of [Z]) boundary condition. The latter can be easily investigated by means of the *polymer expansion* constructed for it in the Pirogov-Sinai theory. This also allows the extension of the uniqueness results from systems with a unique ground state to the case with several ground states but unique *stable ground state* (see [Z]).

Since the publication of the paper [PS] about twenty years ago the Pirogov-Sinai theory was extended in different directions. For a good exposition of the initial theory we refer the reader to [Si] and [Sl]. Some of the generalizations can be found in [BKL], [BS], [DS], [DZ], [HKZ] and [P]. Below we present our results in the standard settings of [PS] and [Z]: a *finite spin space with a translation-periodic finite potential of finite range*, a *finite degeneracy of the ground state* and a *stability of the ground states* expressed via the so called *Peierls* or *Gertzik-Pirogov-Sinai condition.* The extension to other cases is straightforward. Our method also works for unbounded spins, see [LM].

2. Models and Results.

The models are defined on some lattice, which for the sake of simplicity we take to be the d-dimensional ($d \geq 2$) cubic lattice $\mathbf{Z}^d$. The spin variable σ_x associated with the lattice site x takes values from the finite set $S = \{1, 2, \ldots, |S|\}$. The energy of the configuration $\sigma \in S^{\mathbf{Z}^d}$ is given by the formal Hamiltonian

$$H_0(\sigma) = \sum_{A \subset \mathbf{Z}^d,\ \mathrm{diam}\ A \leq r} U_0(\sigma_A). \tag{1}$$

Here $\sigma_A \in S^A$ is a configuration in $A \subset \mathbf{Z}^d$ and the potential $U_0(\sigma_A):\ S^A \mapsto \mathbf{R}$, satisfies $U_0(\sigma_A) = U_0(\sigma_{A+y})$ for any y belonging to some subgroup of $\mathbf{Z}^d$ of finite index and the sum is extended over subsets A of $\mathbf{Z}^d$ with a diameter not exceeding r. Accordingly for a finite domain $V \subset \mathbf{Z}^d$, with the boundary condition $\bar{\sigma}_{V^c}$ given on its complement $V^c = \mathbf{Z}^d \setminus V$, the conditional Hamiltonian is

$$H_0(\sigma_V | \bar{\sigma}_{V^c}) = \sum_{A \cap V \neq \emptyset,\ \mathrm{diam}\ A \leq r} U_0(\sigma_A), \tag{2}$$

where $\sigma_A = \sigma_{A \cap V} + \bar{\sigma}_{A \cap V^c}$ for $A \cap V^c \neq \emptyset$, i.e. the spin at site x is equal to σ_x for $x \in A \cap V$ and $\bar{\sigma}_x$ for $x \in A \cap V^c$.

A *ground state* of (1) is a configuration σ in $\mathbf{Z}^d$ whose energy cannot be lowered by changing σ in some local region. We assume that (1) has a finite number of translation-periodic (i.e. invariant under the action of some subgroup of $\mathbf{Z}^d$ of finite index) ground states. By a standard trick of partitioning the lattice into disjoint cubes $Q(y)$ centered at $y \in q\mathbf{Z}^d$ with an appropriate q and enlarging the spin space from S to S^Q one can

transform the model above into a model on $q\mathbf{Z}^d$ with a *translation-invariant* potential and only *translation-invariant or non-periodic* ground states. Hence, without loss of generality, we assume translation-invariance instead of translation-periodicity and we permute the spin so that the ground states of the model will be $\sigma^{(1)}, \ldots, \sigma^{(m)}$ with $\sigma_x^{(k)} = k$ for any $x \in \mathbf{Z}^d$. Taking $q > r$ one obtains a model with nearest neighbor and next nearest neighbor (diagonal) interaction, i.e. the potential is not vanishing only on lattice cubes Q_1 of linear size 1, containing 2^d sites.

Given a configuration σ in $\mathbf{Z}^d$ we say that site x *is in the k-th phase* if this configuration coincides with $\sigma^{(k)}$ inside the lattice cube $Q_2(x)$ of linear size 2 centered at x. Every connected component of sites not in one of the phases is called a *contour of the configuration* σ. It is clear that for $\sigma = \sigma_{V} + \sigma^{(k)}_{V^c}$ contours are connected subsets of V which we denote by $\tilde{\gamma}_1(\sigma), \ldots, \tilde{\gamma}_l(\sigma)$. The important observation is that the excess energy of a configuration σ with respect to the energy of the ground state $\sigma^{(k)}$ is concentrated along the contours of σ. More precisely,

$$H_0(\sigma_{V}|\sigma^{(k)}_{V^c}) - H_0(\sigma^{(k)}_{V}|\sigma^{(k)}_{V^c}) = \sum_i H_0(\tilde{\gamma}_i(\sigma)), \tag{3}$$

where

$$H_0(\tilde{\gamma}_i(\sigma)) = \sum_{Q_1\colon\, Q_1 \subseteq \tilde{\gamma}_i(\sigma)} \Big(U_0(\sigma_{Q_1}) - U_0(\sigma^{(k)}_{Q_1}) \Big) \tag{4}$$

and the sum is taken over the unit lattice cubes Q_1, containing 2^d sites. The Peierls condition is

$$H_0(\tilde{\gamma}_i(\sigma)) \geq \tau |\tilde{\gamma}_i(\sigma)|, \tag{5}$$

where $\tau > 0$ is an absolute constant and $|\tilde{\gamma}_i(\sigma)|$ denotes the number of sites in $\tilde{\gamma}_i(\sigma)$.

Consider now a family of Hamiltonians $H_n(\sigma) = \sum_{A \subset \mathbf{Z}^d,\ \mathrm{diam}\ A \leq r} U_n(\sigma_A)$, $n = 1, \ldots, m-1$ satisfying the same conditions as H_0 with the same or smaller set of translation-periodic ground states. For $\lambda = (\lambda_1, \ldots, \lambda_{m-1})$ belonging to a neighborhood of the origin in $\mathbf{R}^{m-1}$ define a perturbed formal Hamiltonian

$$H = H_0 + \sum_{n=1}^{m-1} \lambda_n H_n. \tag{6}$$

Here $\lambda_n H_n$ play the role of generalized magnetic fields removing the degeneracy of the ground state. The finite volume Gibbs distribution is

$$\mu_{V,\bar{\sigma}_{V^c}}(\sigma_{V}) = \frac{\exp\left[-\beta H(\sigma_{V}|\bar{\sigma}_{V^c})\right]}{\Xi(V|\bar{\sigma}_{V^c})}, \tag{7}$$

where $\beta > 0$ is the inverse temperature and $\mu_{V,\bar{\sigma}_{V^c}}(\sigma_{V})$ is the probability of the event that the configuration in V is σ_{V}, given $\bar{\sigma}_{V^c}$. Here the conditional Hamiltonian is

$$H(\sigma_{V}|\bar{\sigma}_{V^c}) = \sum_{Q_1\colon\, Q_1 \cap V \neq \emptyset} U(\sigma_{Q_1}), \quad U(\cdot) = \sum_{n=0}^{m-1} U_n(\cdot) \tag{8}$$

and the partition function is

$$\Xi(V|\bar{\sigma}_{V^c}) = \sum_{\sigma_V} \exp\left[-\beta H(\sigma_V|\bar{\sigma}_{V^c})\right] \tag{9}$$

The notion of a *stable ground state* was introduced in [Z] (see also the next section) and it is crucial for the Pirogov-Sinai theory because of the following theorem.

Theorem [PS], [Z]. *Consider a Hamiltonian H of the form (6) satisfying all the conditions above. Then for β large enough, $\beta \geq \beta_0(\lambda)$, every stable ground state $\sigma^{(k)}$ generates a translation-invariant Gibbs state*

$$\mu^{(k)}(\cdot) = \lim_{V\to\mathbf{Z}^d} \mu_{V,\sigma^{(k)}_{V^c}}(\cdot). \tag{10}$$

These Gibbs states are different for different k and they are the only translation-periodic Gibbs states of the system.

Remark. Given H_n, $n = 0, \ldots, m-1$ there exists sufficiently small λ_0 such that for $\max_{1\leq n\leq m-1} |\lambda_n| \leq \lambda_0$ the quantity β_0 becomes independent on λ.

An obvious corollary of the above theorem is

Corollary. *If there is only a single stable ground state, say $\sigma^{(1)}$, then for $\beta \geq \beta_0(\lambda)$ there is a unique translation-periodic Gibbs state*

$$\mu^{(1)}(\cdot) = \lim_{V\to\mathbf{Z}^d} \mu_{V,\sigma^{(1)}_{V^c}}(\cdot). \tag{11}$$

Our extension of this result is given by

Theorem. *Consider a Hamiltonian H of the form (6) satisfying all the conditions above. Suppose that given λ and $\beta \geq \beta_0(\lambda)$ there exist a single stable ground state, say $\sigma^{(1)}$. Then the Gibbs state $\mu(\cdot) = \mu^{(1)}(\cdot)$ is unique. Moreover, for finite $A \subset V$, such that* $\mathrm{dist}(A, V^c) \geq 2d\,\mathrm{diam}A + C_1(\tau, \beta, \lambda, d)$, *any configuration σ_A and any boundary condition $\bar{\sigma}_{V^c}$, one has*

$$\left|\mu_{V,\bar{\sigma}_{V^c}}(\sigma_A) - \mu(\sigma_A)\right| \leq \exp\left[-C_2(\tau, \beta, \lambda, d)\ \mathrm{dist}\ (A, V^c)\right], \tag{12}$$

where $C_1, C_2 > 0$.

Remark. In contrast with [DP], [Sh] and [M1,2] the theorem above treats the situation when there are several ground states with only one of them being stable. Moreover, the result is true for all sufficiently low temperatures not depending on how close the parameters are to the points with non unique ground state. If some of the conditions of the Theorem are violated the statement can be wrong.

The simplest counterexample can be constructed from the Ising model in $d = 3$. It is well-known that at low temperatures this model contains precisely two translation-invariant Gibbs states taken into each other by $\pm$ symmetry and infinitely many non

translation-invariant Gibbs states, i.e. the Dobrushin states mentioned earlier. Identifying configurations taken into each other by $\pm$ symmetry one obtains a model with unique translation-invariant Gibbs state and infinitely many non translation-invariant ones. The condition of the Theorem which is not true for this factorized model is the finiteness of the potential: the model contains a hard-core constraint.

Formally speaking, one can consider a model with spin variables still taking values ± 1 but assigned to bonds of the lattice. Then the Hamiltonian is the sum of the spins over all lattice bonds multiplied by a negative coupling constant. The hard-core constraint says that the product of spins along any lattice plaquette is 1.

Another example based on a gauge model can be found in [B]. For this model the spin space is finite, the potential is finite and of finite radius but the Peierls condition is violated and the number of ground states is infinite. On the other hand, after a proper factorization, this system can be transformed into a model with a hard-core restriction similar to that discussed above.

It is known [LML] that for systems satisfying FKG inequalities the uniqueness of the translation-invariant Gibbs state implies uniqueness of all Gibbs states at any temperature. This makes it tempting to assume that the conclusions of our theorem hold beyond the low temperature region covered by the Pirogov-Sinai theory. We are not aware of any counterexample.

3. Proof of the Theorem

In this section we assume that the reader is familiar with the Pirogov-Sinai theory and we only list the appropriate notation. Then we quote some necessary results and proceed to the proof of the theorem.

Preliminaries. A *contour* is a pair $\gamma = (\tilde{\gamma}, \sigma_{\tilde{\gamma}})$ consisting of the *support* $\tilde{\gamma}$ and the configuration $\sigma_{\tilde{\gamma}}$ in it. The components of the *interior* of the contour γ are denoted by $Int_j\gamma$ and the *exterior* of γ is denoted by $Ext\gamma$. The family $\{\tilde{\gamma}, Int_j\gamma, Ext\gamma\}$ is a partition of $\mathbf{Z}^d$. The configuration $\sigma_{\tilde{\gamma}}$ can be uniquely extended to the configuration σ' in $\mathbf{Z}^d$ taking constant values $I_j(\gamma)$ and $E(\gamma)$ on the connected components of $\tilde{\gamma}^c$. Generally these values are different for different components. The contour γ is said to be *from the phase* k if $E(\gamma) = k$. The *energy* of the contour γ is

$$H(\tilde{\gamma}(\sigma)) = \sum_{Q_1:\; Q_1 \subseteq \tilde{\gamma}(\sigma)} \Big(U(\sigma_{Q_1}) - U(\sigma_{Q_1}^{(E(\gamma))}) \Big) . \tag{13}$$

The *statistical weight* of γ is

$$w(\gamma) = \exp(-\beta H(\gamma)) \tag{14}$$

and satisfies

$$0 \le w(\gamma) \le e^{-\beta\tau|\tilde{\gamma}|}. \tag{15}$$

The *renormalized statistical weight* of the contour is

$$W(\gamma) = w(\gamma) \prod_j \frac{\Xi(Int_j^* \gamma | I_j(\gamma))}{\Xi(Int_j^* \gamma | E(\gamma))}, \tag{16}$$

where for any $A \subset \mathbf{Z}^d$ we denote

$$A^* = \{x \in A | \ x \ \text{ is not adjacent to } A^c\}. \tag{17}$$

The contour γ is *stable* if

$$W(\gamma) \leq \exp\left[-\frac{1}{3}\beta\tau|\tilde{\gamma}|\right] \tag{18}$$

and the ground state $\sigma^{(k)}$ is *stable* if all contours γ with $E(\gamma) = k$ are stable. It is known (see [Z]) that at least one of the ground states is stable. Because of (18) for any $x \in \mathbf{Z}^d$, $N \geq 1$ and β large enough

$$\sum_{\gamma:\ (Ext\gamma)^c \ni x,\ |\tilde{\gamma}| \geq N} W(\gamma) \leq e^{-C_3 N}, \tag{19}$$

where $C_3 = C_3(\tau, \beta, d)$ is positive and monotone increasing in τ and β. In particular

$$\sum_{\gamma:\ (Ext\gamma)^c \ni x} W(\gamma) \leq C_4, \tag{20}$$

where $C_4 = e^{-C_3}$. For the stable ground state $\sigma^{(k)}$ the corresponding partition function can be represented as

$$\Xi(V|k) = e^{-\beta H(\sigma_V^{(k)} | \sigma_{V^c}^{(k)})} \sum_{[\gamma_i]^s \in V,\ E([\gamma_i]^s) = k} \prod_i W(\gamma_i), \tag{21}$$

where the sum is taken over all collections of contours $[\gamma_i]$ such that $\tilde{\gamma}_i$ are disjoint, $E(\gamma_i) = k$ for all i and $\tilde{\gamma}_i \subseteq V$ for all i.

Representation (21) and estimate (18) allow to write an absolutely convergent polymer expansion

$$\log \Xi(V|k) = -\beta H(\sigma_V^{(k)} | \sigma_{V^c}^{(k)}) + \sum_{\pi^{(k)} \in V} W(\pi^{(k)}), \tag{22}$$

where the sum is taken over so called *polymers* $\pi^{(k)}$ of the phase k belonging to the domain V. By definition a polymer $\pi^{(k)} = (\gamma_i)$ is a collection of, not necessarily different, contours γ_i of the phase k such that $\cup_i \tilde{\gamma}_i$ is connected. The statistical weight $W(\pi^{(k)})$ is uniquely defined via $W(\gamma_i)$ and satisfies the estimate (see [Se])

$$|W(\pi^{(k)})| \leq \exp\left[-\left(\frac{1}{3}\beta\tau - 6d\right) \sum_i |\tilde{\gamma}_i|\right] \tag{23}$$

implying

$$\sum_{\pi^{(k)}=(\gamma_i):\, \cup_i (Ext\gamma_i)^c \ni x,\, \sum_i |\tilde{\gamma}_i| \geq N} |W(\pi^{(k)})| \leq e^{-C_3 N}. \tag{24}$$

Denote by $\mu_V^{(k)}(\{\gamma_i\}, ext)$ the probability of the event that all contours of the collection $\{\gamma_i\}$ are external ones inside V. By the construction

$$\mu_V^{(k)}(\{\gamma_i\}, ext) \leq \prod_i W(\gamma_i). \tag{25}$$

From the polymer expansion (22) and estimate (24) it is not hard to conclude that for $\{\gamma_i\}$ with $\text{dist}(\cup_i \tilde{\gamma}_i, V^c) \geq | \cup_i \tilde{\gamma}_i|$

$$\left|\mu_V^{(k)}(\{\gamma_i\}, ext) - \mu^{(k)}(\{\gamma_i\}, ext)\right| \leq \mu^{(k)}(\{\gamma_i\}, ext) | \cup_i \tilde{\gamma}_i| \exp\left[-C_5 \text{ dist } (\cup_i \tilde{\gamma}_i, V^c)\right], \tag{26}$$

where $C_5 = C_5(\tau, \beta, d)$ is positive and monotone increasing in τ and β. For any $A \in V$, $\text{dist}(A, V^c) \geq \text{diam}A$, and any σ_A estimate (26) implies in a standard way that

$$\left|\mu_V^{(k)}(\sigma_A) - \mu^{(k)}(\sigma_A)\right| \leq \exp\left[-C_6 \text{ dist } (A, V^c)\right], \tag{27}$$

where again $C_6 = C_6(\tau, \beta, d)$ is positive and monotone increasing in τ and β.

For an arbitrary boundary condition $\bar{\sigma}_{V^c}$ the probability $\mu_{V, \bar{\sigma}_{V^c}}(\sigma_A)$ depends only on $\bar{\sigma}_{\partial V}$, where

$$\partial V = \{x \in V^c :\; x \text{ is adjacent to } V\}, \tag{28}$$

and we freely use the notation $\mu_{V, \bar{\sigma}_{\partial V}}(\sigma_A)$. From now on we suppose that $\sigma^{(1)}$ is the only stable ground state of H and denote by $\mu_V(\cdot)$ the Gibbs distributions with the stable boundary condition $\sigma^{(1)}_{V^c}$.

For a domain V with the boundary condition $\sigma^{(1)}_{V^c}$ fix $l \leq L$ sites belonging to ∂V and consider a collection $\{\gamma_i\}^e \in V$ of external contours touching ∂V at one of these sites. A smaller domain $V' = \cup_i \cup_j Int_j \gamma_i$ has a natural boundary condition $\sigma'_{\partial V'}$ induced by $\cup_i \sigma_{\tilde{\gamma}_i}$. Given $M \geq \sum_i |\tilde{\gamma}_i|$ denote by $\mathcal{E}_{\{\gamma_i\}^e, M}$ the event that the total number of sites in all adjacent to $\partial V'$ connected components of sites not in the 1-st phase is not less than $M - \sum_i |\tilde{\gamma}_i|$. According to the Theorem of Section 3.2 in [Z]

$$\mu_{V', \sigma'_{\partial V'}}\left(\mathcal{E}_{\{\gamma_i\}^e, M}\right) \leq e^{-C_7(\tau, \beta, \lambda, d)(M - \sum_i |\tilde{\gamma}_i|) + C_8(\tau, \beta, \lambda, d) \sum_i |\tilde{\gamma}_i|}. \tag{29}$$

The positive constants C_7 and C_8 tend to 0 as $\beta \to \infty$ or (β, λ) approaches the manifold on which $\sigma^{(1)}$ is not the only stable ground state. For different $\{\gamma_i\}^e$ the events $\mathcal{E}_{\{\gamma_i\}^e, M}$ are disjoint and for their union $\mathcal{E}_{V, M, L} = \cup_{\{\gamma_i\}^e \in V} \mathcal{E}_{\{\gamma_i\}^e, M}$ one has the estimate

$$\mu_V\left(\mathcal{E}_{V, M, L}\right) = \sum_{\{\gamma_i\}^e \in V} \mu_{V', \sigma'_{\partial V'}}\left(\mathcal{E}_{\{\gamma_i\}^e, M}\right) \mu_V\left(\{\gamma_i\}^e\right)$$

$$
\begin{aligned}
&\leq \sum_{\{\gamma_i\}^e \in V} e^{-C_7(M-\sum_i |\tilde{\gamma}_i|)+C_8|\tilde{\gamma}_i|} \prod_i W(\gamma_i) \\
&\leq e^{-C_7 M} \sum_{\{\gamma_i\}^e \in V} \prod_i e^{-\left(\frac{1}{3}\tau\beta - C_7 - C_8\right)|\tilde{\gamma}_i|} \\
&\leq e^{-C_7 M}(1+C_4)^L \\
&\leq e^{-C_7 M + C_4 L}. \qquad (30)
\end{aligned}
$$

Finally observe that for any $A \subseteq V$ and any σ_A

$$
\mu_V(\sigma_A) e^{-C_9 L(\bar{\sigma}_{\partial V})} \leq \mu_{V,\bar{\sigma}_{V^c}}(\sigma_A) \leq \mu_V(\sigma_A) e^{C_9 L(\bar{\sigma}_{\partial V})}, \qquad (31)
$$

where $L(\bar{\sigma}_{\partial V})$ is the number of sites $x \in \partial V$ with $\bar{\sigma}_{\partial V} \neq 1$ and

$$
C_9 = 2^d \beta \max_{\sigma_{Q_1}} |U(\sigma_{Q_1})| . \qquad (32)
$$

Proof. We are now ready to prove the theorem. Take an integer $N > 0$ and suppose that V contains a cube Q_{6N} with sides of length 6N centered at the origin. From now on all cubes are assumed to be centered at the origin. Let $Q_{N'}$, $N' \geq 6N$ be the maximal cube contained in V. Denote $\partial' V = \partial Q_{N'} \cap \partial V$. First we consider boundary conditions $\bar{\sigma}_{\partial V}$ which coincide with $\sigma^{(1)}$ on $\partial V \setminus \partial' V$ and differ from $\sigma^{(1)}$ on $\partial' V$ by at most $\sqrt{N}$ lattice sites.

Given σ_V denote by $\Omega(\sigma_V)$ the union of the connected components of the set $\{x \in V :\ x \text{ is not in the 1-st phase } \}$ adjacent to $\{x \in \partial V :\ \bar{\sigma}_x \neq 1\}$. This set is called the *boundary layer* of σ_V and we denote by $\Omega_i(\sigma_V)$ its connected components. Introduce the event $\mathcal{E}_0 = \{\sigma_V :\ \Omega(\sigma_V) \cap Q_{4N} \neq \emptyset\}$. By construction for $\sigma_V \in \mathcal{E}_0$ every $\Omega_i(\sigma_V)$ touches ∂V at some site $x \in \partial V$ with $\bar{\sigma}_x \neq 1$ and there exists at least one component Ω_i intersecting Q_{4N}. Without loss of generality we suppose that it is $\Omega_1(\sigma_V)$. This leads to the estimate

$$
\begin{aligned}
\mu_{V,\bar{\sigma}_{V^c}}(\mathcal{E}_0) &\leq e^{C_9\sqrt{N}} \mu_V(\mathcal{E}_0) \\
&\leq e^{C_9\sqrt{N}} \sqrt{N} e^{-C_3 N} (1+C_4)^{\sqrt{N}} \\
&\leq e^{-C_{10} N} . \qquad (33)
\end{aligned}
$$

In the first inequality of (33) we used (31) reducing the problem to the calculation for the stable boundary condition $\sigma^{(1)}$. The second inequality comes in a standard way from the cluster expansion for $\mu_V(\cdot)$. Indeed, in the domain V with the stable boundary condition $\sigma^{(1)}_{V^c}$ every component Ω_i contains an external contour γ_i such that $E(\gamma_i) = 1$, $\tilde{\gamma}_i \subseteq \Omega_i$ and $\Omega_i \subseteq (Ext(\gamma_i))^c$. One may simply say that γ_i is the external boundary of Ω_i and clearly $\tilde{\gamma}_i$ touches ∂V. If Ω_1 intersects Q_{4N} then $\tilde{\gamma}_1$ intersects or encloses Q_{4N}. The number of possibilities to chose the site $x \in \partial V$, $\bar{\sigma}_x \neq 1$ at which $\tilde{\gamma}_1$ touches ∂V

does not exceed $\sqrt{N}$ which produces the factor $\sqrt{N}$ in the estimate. The next factor estimates the sum of the statistical weights of all possible γ_1 touching this site. It is based on (19) and takes into account the fact that the diameter of $\tilde{\gamma}_1$, and hence $|\tilde{\gamma}_1|$, is not less than N. The constant C_4 (see (20)) estimates the sum of the statistical weights of all possible γ_i touching given lattice site and $(1+C_4)^{\sqrt{N}}$ estimates the statistical weight of all possibilities to choose $\{\gamma_i,\ i \neq 1\}$. The whole estimate uses (25) and the fact that $\mu_V(\{\gamma_i\}, ext)$ is the upper bound for the sum of μ_V-probabilities of boundary layers $\Omega = \{\Omega_i\}$ having γ_i as the boundary of Ω_i. The third inequality in (33) is trivial for $C_{10} = 0.5C_3$ and $N \geq 4(C_9+1+C_4)^2 C_3^{-2}$.

Denote by $\mathcal{E}_0^c$ the complement of $\mathcal{E}_0$. If $\sigma_V \in \mathcal{E}_0^c$ then $(V \setminus \Omega(\sigma_V))^* \supseteq Q_{4N-2}$ (see (17) for the definition of $(\cdot)^*$). It is not hard to see that the configuration $\sigma_V \in \mathcal{E}_0^c$ equals 1 on the boundary of $(V \setminus \Omega(\sigma_V))^*$. Now fix $A \subset Q_{2N}$ and σ_A. In view of (27) one has

$$\left|\mu_{V,\bar{\sigma}_{V^c}}(\sigma_A|\mathcal{E}_0^c) - \mu(\sigma_A)\right| \leq e^{-C_6 N}. \tag{34}$$

This gives us

$$\begin{aligned}\left|\mu_{V,\bar{\sigma}_{V^c}}(\sigma_A) - \mu(\sigma_A)\right| &\leq \left|\mu_{V,\bar{\sigma}_{V^c}}(\sigma_A|\mathcal{E}_0) - \mu(\sigma_A)\right| \mu_{V,\bar{\sigma}_{V^c}}(\mathcal{E}_0) \\ &\quad + \left|\mu_{V,\bar{\sigma}_{V^c}}(\sigma_A|\mathcal{E}_0^c) - \mu(\sigma_A)\right| \mu_{V,\bar{\sigma}_{V^c}}(\mathcal{E}_0^c) \\ &\leq e^{-C_{10}N} + e^{-C_6 N} \\ &\leq e^{-C_{11}N}\ ,\end{aligned} \tag{35}$$

where $C_{11} = 0.5\min(C_{10}, C_6)$ and $N \geq \log 2/C_{11}$.

To extend (35) to the wider class of boundary conditions we suppose that $V \supseteq Q_{8N}$ and $Q_{N'}$, $N' \geq 8N$ is the maximal cube contained in V. Now we consider boundary conditions $\bar{\sigma}_{\partial V}$ which coincide with $\sigma^{(1)}$ on $\partial V \setminus \partial' V$ and differ from $\sigma^{(1)}$ on $\partial' V$ by at most $(\sqrt{N})^2$ lattice sites. Denote $\Delta_i = Q_{8N-2i+2} \setminus Q_{8N-2i}$ and let $\Omega^{(i)}(\sigma_V)$ be a union of the connected components of the set $\{x \in V \setminus Q_{8N-2i} :\ \sigma_x \neq 1\}$ adjacent to $\{x \in \partial V :\ \bar{\sigma}_x \neq 1\}$, $i = 1, \ldots, N$. Introduce disjoint events

$$\begin{aligned}\mathcal{E}_1 &= \{\sigma_V : |\Omega^{(1)}(\sigma_V) \cap \Delta_1| < \sqrt{N}\}, \\ \mathcal{E}_i &= \{\sigma_V : |\Omega^{(1)}(\sigma_V) \cap \Delta_1| \geq \sqrt{N}, \ldots, |\Omega^{(i-1)}(\sigma_V) \cap \Delta_{i-1}| \geq \sqrt{N}, \\ &\qquad |\Omega^{(i)}(\sigma_V) \cap \Delta_i| < \sqrt{N}\}\end{aligned}$$

and

$$\mathcal{E}_c = \left(\bigcup_{i=1}^{N} \mathcal{E}_i\right)^c . \tag{36}$$

If $\sigma_V \in \mathcal{E}_c$ than the boundary contour $\Omega(\sigma_V)$ contains at least $N\sqrt{N}$ sites. Hence (30) implies the following estimate for the probability of $\mathcal{E}_c$

$$\begin{aligned}\mu_{V,\bar{\sigma}_{V^c}}(\mathcal{E}_c) &\leq e^{C_9(\sqrt{N})^2} \mu_V(\mathcal{E}_c) \\ &\leq e^{C_9(\sqrt{N})^2} e^{-C_7 N\sqrt{N} + C_4(\sqrt{N})^2} \\ &\leq e^{-C_{12}N\sqrt{N}}\ ,\end{aligned} \tag{37}$$

where $C_{12} = 0.5C_7$ and $N \geq 4(C_9 + C_4)^2 C_7^{-2}$.

For $\sigma_V \in \mathcal{E}_i$ consider the volume $V_i = (V \setminus \Omega^{(i)}(\sigma_V))^* \cup Q_{8N-2i}$ with the boundary condition $\bar{\sigma}_{V^c} + \sigma_{V \setminus V_i}$. By construction the number of sites $x \in \partial V_i$ with $\sigma_x \neq 1$ is less than $\sqrt{N}$ and one can apply (35) to obtain the bound

$$\Big| \mu_{V, \bar{\sigma}_{V^c}}(\sigma_A | \mathcal{E}_i) - \mu(\sigma_A) \Big| \leq e^{-C_{11}N} . \tag{38}$$

Joining (37) and (38) we conclude

$$\begin{aligned} \Big| \mu_{V, \bar{\sigma}_{V^c}}(\sigma_A) - \mu(\sigma_A) \Big| &= \Bigg| \sum_{i=1}^{N} \mu_{V, \bar{\sigma}_{V^c}}(\mathcal{E}_i) \Big(\mu_{V, \bar{\sigma}_{V^c}}(\sigma_A | \mathcal{E}_i) - \mu(\sigma_A) \Big) \\ &\quad + \mu_{V, \bar{\sigma}_{V^c}}(\mathcal{E}_c) \Big(\mu_{V, \bar{\sigma}_{V^c}}(\sigma_A | \mathcal{E}_c) - \mu(\sigma_A) \Big) \Bigg| \\ &\leq e^{-C_{11}N} + e^{-C_{12}N\sqrt{N}} \\ &\leq 2e^{-C_{11}N} , \end{aligned} \tag{39}$$

where $N \geq (C_{11}/C_{12})^2$. Expression (39) is a version of (35) which is weaker by the factor 2 in the RHS but is applicable to the wider class of boundary conditions containing $(\sqrt{N})^2$ unstable sites instead of $\sqrt{N}$ for (35).

The argument leading from (35) to (39) can be iterated several times. The first iteration treats the following situation. Suppose that $V \supseteq Q_{10N}$ and $Q_{N'}$, $N' \geq 10N$ is the maximal cube contained in V. Consider boundary conditions $\bar{\sigma}_{\partial V}$ which coincide with $\sigma^{(1)}$ on $\partial V \setminus \partial' V$ and differ from $\sigma^{(1)}$ on $\partial' V$ by at most $(\sqrt{N})^3$ lattice sites. Then the analogue of (39) is

$$\Big| \mu_{V, \bar{\sigma}_{V^c}}(\sigma_A) - \mu(\sigma_A) \Big| \leq 3e^{-C_{11}N}. \tag{40}$$

Similarly after $2d$ iterations one obtains that for any $V \supseteq Q_{2dN}$ with the boundary condition $\bar{\sigma}_{\partial V}$ containing not more than $(\sqrt{N})^{2d}$ unstable sites

$$\Big| \mu_{V, \bar{\sigma}_{V^c}}(\sigma_A) - \mu(\sigma_A) \Big| \leq 2de^{-C_{11}N}. \tag{41}$$

Now set $C_1 = 2d \max\big(4(C_9 + 1 + C_4)^2 C_3^{-2},\ \log 2/C_{11},\ 4(C_9 + C_4)^2 C_7^{-2},\ (C_{11}/C_{12})^2\big)$ and for any A and σ_A consider a cube Q_{2L} with $L \geq C_1$ and $\mathrm{dist}(A, Q_{2L}^c) \geq (1 - 1/d)L$. Taking $N = L/d$ and $C_2 = C_{11}/2d$ for $V = Q_{2L}$ one obtains (12) from (41). For any $V \supseteq Q_{2L}$ with $\partial V \cap \partial Q_{2L} \neq \emptyset$ we have

$$\begin{aligned} \Big| \mu_{V, \bar{\sigma}_{V^c}}(\sigma_A) - \mu(\sigma_A) \Big| &\leq \sum_{\sigma_{\partial Q_{2L}}} \Big| \mu_{Q_{2L}, \sigma_{\partial Q_{2L}}}(\sigma_A) - \mu(\sigma_A) \Big| \mu_{V, \bar{\sigma}_{V^c}}(\sigma_{\partial Q_{2L}}) \\ &\leq \exp\left[-C_2 \text{ dist } (A, V^c)\right] , \end{aligned} \tag{42}$$

which finishes the proof of the **Theorem**.

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